HANDOUT · BIRTHDAY PARADOX

DANIEL BILAR
DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF NEW ORLEANS

Summary

Why is it that you only need 23 people in a room to have a 50%+ chance for any two to share the same birthday? In a room of 75 it's a 99.9% chance of any two people matching. The answer lies in the growth rate of pair-wise comparisons between n people which grows as $O(n^2)$.

1. One intuition: Count the pairings

If you have 3 people - say Aaron, Barrett, Chris - there are 3 possibilities of them sharing birthdays (A-B, B-C, A-C). With 4 people, there are 6, with 5 there are 10.

- In general for n people, the combination formula is $C(n, 2) = \frac{n!}{(n-2)!k!} = \frac{n(n-1)}{2}$. For 23 people it gives you $\frac{23*22}{2} = 253$ pairs.
- Now, the probability of just 2 people - this is 1 pair - have different b-days is $1 - \frac{1}{365} = \frac{364}{365} = 0.99726$
- For 253 pairs (with the approximation that the birthdays are all independent which is not true, but close enough), the probability of all of them having different b-days is $(\frac{364}{365})^{253} = 0.49952284$.

2. Second way: Doing the math semi-exact

Two people have, as stated above, a $(1 - \frac{1}{365}) = 99.7\%$ chance of different b-day. Three people have a probability p(different) $= (1 - \frac{1}{365})(1 - \frac{2}{365})$, since the third person has to have a b-day different from the first *and* the second person (this we disregarded with our simplifying approximation of each bday being independent in the previous section 1).

- So for 23 people to have a different b-day, we have $p(different) = (1 - \frac{1}{365})(1 - \frac{2}{365}) \cdot\cdot (1 - \frac{22}{365})$
- Now, we use a little math trick: $e^x \approx 1 + x$ when x is close to 0 .. so we have $(1 - \frac{1}{365}) \approx e^{-\frac{1}{365}}$
- We rewrite $p(different) = (1 - \frac{1}{365})(1 - \frac{2}{365}) \cdot\cdot (1 - \frac{22}{365})$ as
  $= e^{-\frac{1}{365}} e^{-\frac{2}{365}} \cdot\cdot e^{-\frac{22}{365}}$
  $= e^{-\frac{1+2+..+22}{365}}$
- Now adding numbers from 1 to y - math genius Gauss figured this out when he was 8 years old in the 18th century - is $y = \frac{y(y+1)}{2}$, so adding 1 up to 22 is $\frac{22*23}{2}$
- Plugging it all in, we get $p(different) = e^{-\frac{23*22}{2*365}} = 0.499998248$.

  You see that the approximation in section 1 is very close 0.49952284 - which is why it sometimes pays not to be too exact and can trade for simplicity.